PERGAMON

Applied Mathematics Letters 14 (2001) 307–311

Applied Mathematics Letters

www.elsevier.nl/locate/aml

A Characterization of a Family of Consensus Rules for Committee Elections

H.-H. BOCK
Institut für Statistik, Technische Hochschule Aachen
D-52056 Aachen, Germany

W. H. E. DAY
P.O. Box 17, Port Maitland, NS B0W 2V0, Canada

F. R. McMORRIS*
Department of Applied Mathematics, Illinois Institute of Technology
Chicago, IL 60616, U.S.A.
mcmorris@iit.edu

(Received March 2000; accepted April 2000)

Communicated by F. Harary

Abstract—A committee election rule is proposed that is based on a measure of vote concentration. Relationships are established between properties of the committee selection function, and properties of the function used to measure vote concentration.

Keywords—Committee election, Consensus rule, Vote concentration.

1. INTRODUCTION

Suppose there is a set of n voters, whose task is to select a committee from a nonempty set $S = \{s_1, \dots, s_m\}$ of m candidates. Each voter is to vote for exactly one candidate, and we assume that the committee elected depends only on the number of votes cast for each candidate. Thus, any committee elected from S depends only on the proportions α_i of votes cast for each candidate s_i, $i = 1, \dots, m$. Without loss of generality, we will assume that S has been relabeled so that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$. Therefore, the result of the ballot can be represented as a partition of 1 into m nonnegative parts, $\alpha = (\alpha_1, \dots, \alpha_m)$, with $\alpha_1 \geq \cdots \geq \alpha_m \geq 0$ and $\sum_{i=1}^m \alpha_i = 1$. For example,

$$\alpha = \left(\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, 0, 0, 0\right)$$

depicts an election in which (a multiple of) 12 votes were cast, three candidates receiving no votes.

The work of the second and third authors was partially supported by the Office of Naval Research Grant N00014-00-1-0004.

*Author to whom all correspondence should be addressed.

PII: S0893-9659(00)00154-3

In previous papers [1,2], we proposed a committee selection method which was motivated by a consensus approach used in biological sequence analysis [3–5]. This method uses the idea of vote concentration to give a committee election rule that returns optimal sizes (and composition) of committees that meet the conditions, so that the committee size is not specified before the election.

In the present note, we investigate a relationship between the properties of the committee selection function C, and properties of the function f which is used for measuring vote concentration. In fact, we find that C fulfills the properties of "minimal eligibility" and "extreme neutrality" (to be defined below) if and only if f is increasing and strictly concave downward.

2. CONSENSUS RULES

Let $\Pi(X)$ denote the set of all nonempty subsets of a set X, and P_m the set of all partitions of 1 into m nonnegative parts. We will model an election procedure described above as a consensus function of the form

$$C : P_m \to \Pi(M),$$

where $M = \{1, \ldots, m\}$. Thus, the procedure returns a set of committee sizes for each partition of 1. In fact, a relevant committee of size k will always consist of the k "first" candidates (i.e., those with the largest proportions α_i), and therefore, it is sufficient to specify the set $C(\alpha)$ of integers in $\{1, \ldots, m\}$ which are selected as committee sizes. If $|C(\alpha)| > 1$, then our method returns more than one committee, and other criteria (e.g., favoring small or large committee size) might need to be applied to chose among them.

As motivation for the general type of committee election rules that we proposed, consider the analog of the *simple plurality rule*. In our setting, this is the function

$$\text{spl} : P_m \to \Pi(M),$$

such that, for all $\alpha = (\alpha_1, \ldots, \alpha_m) \in P_m$ and $x \in M$,

$$x \in \text{spl}(\alpha) \iff \frac{\sum_1^x \alpha_i}{x} \geq \frac{\sum_1^j \alpha_i}{j}, \qquad \text{for all } j \in M, \tag{1}$$

and x is maximum with respect to this property, so that $\text{spl}(\alpha) = \{x\}$.

In general, if x is returned as the size of an elected committee, it is natural to expect that the votes are highly concentrated on the members of that committee. As can be seen above, spl uses an appealing measure of vote concentration: the average number of votes per committee member. There are some possibly adverse consequences of this choice, however. Consider the following example. If $\alpha = (0.997, 0.001, 0.001, 0.001)$ and $\gamma = (0.253, 0.249, 0.249, 0.249)$, then $\text{spl}(\alpha) = \{1\} = \text{spl}(\gamma)$, and we see that the spl's result for γ fails to indicate that unelected committees of sizes 2, 3, and 4 have essentially the same vote concentrations as the elected committee of size 1. Perhaps a committee of size x should be returned only if its vote concentration significantly exceeds the vote concentrations of committees of other sizes. To do this, we let $f : M \to \Re^{>0}$ be a measure of vote concentration, where $\Re^{>0}$ is the set of positive real numbers, and use f to affect vote concentration by transforming the committee sizes found in the denominators of (1).

We will now investigate the relationships between f, which is interpreted as a measure of vote concentration, and a corresponding consensus rule C_f, which is defined by generalizing (1).

DEFINITION 2.1. *Let $f : M \to \Re^{>0}$ be a measure of vote concentration. The consensus rule associated with f is the consensus function*

$$C_f : P_m \to \Pi(M),$$

such that, for all $\alpha = (\alpha_1, \ldots, \alpha_m) \in P_m$ *and* $x \in M$,

$$x \in C_f(\alpha) \iff \frac{\sum_1^x \alpha_i}{f(x)} \geq \frac{\sum_1^j \alpha_i}{f(j)}, \qquad \text{for all } j \in M. \tag{2}$$

To simplify notation, define $a \div f(j) = (\sum_1^j \alpha_i)/f(j)$, so that (2) becomes

$$x \in C_f(\alpha) \iff a \div f(x) \geq a \div f(j), \qquad \text{for all } j \in M.$$

To delimit in $\alpha \in P_m$ all parts of the same size, define, for all $j \in M$,

$$\begin{aligned} \lambda_j(\alpha) &= \min\{i \in M : \alpha_i = \alpha_j\}, \qquad \text{and} \\ \rho_j(\alpha) &= \max\{i \in M : \alpha_i = \alpha_j\}. \end{aligned}$$

3. AXIOMS

We will now investigate axiomatically the C_f consensus rules of Definition 2.1, so as to characterize a superfamily of the ras_p family of consensus rules which were defined and studied in [1]. Recall that, for any $p \in \Re^{>0}$, ras_p is the function C_{f_p} where the concentration function is specified by $f_p(x) = \sqrt[p]{x}$ for all $x \in M$ (see [1, p. 223]). The function f_p was picked as a simple example of one satisfying desired properties of monotonicity (SIN) and concavity (SCD), which we recall below.

(SIN). *$f : M \to \Re^{>0}$ is strictly increasing in M if*

$$x, y \in M \text{ and } x < y \implies f(x) < f(y).$$

(SCD). *$f : M \to \Re^{>0}$ is strictly concave downward in M if*

$$x, y \in M,\ x \neq y, \text{ and } (x+y) \bmod 2 = 0 \implies f\left(\frac{x+y}{2}\right) > \frac{f(x)+f(y)}{2}.$$

Desirable properties that the C_f rules might satisfy are the following.

EXTREME NEUTRALITY (EN). *Candidates on committees should have more votes than candidates not on committees, i.e.,*

$$x \in C_f(\alpha) \text{ and } x < m \implies \alpha_x > \alpha_{x+1},$$

or, equivalently,

$$x \in C_f(\alpha) \implies x = \rho_x(\alpha).$$

MINIMAL ELIGIBILITY (ME). *Candidates without votes should not be on committees, i.e.,*

$$x \in C_f(\alpha) \implies \alpha_x > 0.$$

NONTRIVIALITY (NT). *Nontrivial consensus rules should have nontrivial results, i.e.,*

$$m > 1 \iff |\{C_f(\alpha) : \alpha \in P_m\}| > 1.$$

The dependencies among these properties yield a characterization of an interesting family of consensus rules.

THEOREM 3.1. *Let $f : M \to \Re^{>0}$ and $C_f : P_m \to \Pi(M)$, the consensus rule associated with f. C_f satisfies (EN) and (ME) if and only if f satisfies (SIN) and (SCD).*

PROOF. Assume that C_f satisfies (EN) and (ME). Then (SIN) is vacuously satisfied for $m = 1$, so let $m > 1$. For any $j \in M \setminus \{m\}$, define $\gamma^j = (\gamma_1, \dots, \gamma_m) \in P_m$ such that

$$\gamma_i = \begin{cases} \dfrac{1}{j}, & \text{for } 1 \le i \le j, \\ 0, & \text{for } j < i \le m. \end{cases} \tag{3}$$

(EN) implies that $C_f(\gamma^j) \subseteq \{j, m\}$, and (ME) implies that $m \notin C_f(\gamma^j)$, so $C_f(\gamma^j) = \{j\}$. Thus, $1/f(j) = \gamma^j \div f(j)$ and $\gamma^j \div f(j+1) = 1/f(j+1)$, and from (2),

$$\frac{1}{f(j)} = \gamma^j \div f(j) > \gamma^j \div f(j+1) = \frac{1}{f(j+1)},$$

so $f(j) < f(j+1)$. It follows that $f(1) < \cdots < f(m)$ and f satisfies (SIN).

Now assume that f does not satisfy (SCD) and pick $\ell, r \in M$ such that $\ell < r$, $(\ell + r) \bmod 2 = 0$, and

$$f\left(\frac{\ell + r}{2}\right) \le \frac{f(\ell) + f(r)}{2}.$$

Without loss of generality, we may assume that $f(\ell) = 0$, $\ell = 0$, and r is even with

$$f\left(\frac{r}{2}\right) \le \frac{f(r)}{2}.$$

Use (EN) and (ME) to define $\gamma^r \in P_m$ by (3), with $j = r$, so that $C_f(\gamma^r) = \{r\}$. Note that

$$\gamma^r \div f(x) = \frac{x}{rf(x)}, \qquad \text{for all } x \in \{1, \dots, r\}.$$

Since $1 \le r/2 < r$ and $r/2 \notin C_f(\gamma^r)$,

$$\frac{1}{2f(r/2)} = \gamma^r \div f\left(\frac{r}{2}\right) < \gamma^r \div f(r) = \frac{1}{f(r)},$$

whence

$$\frac{f(r)}{2} < f\left(\frac{r}{2}\right) \le \frac{f(r)}{2},$$

a contradiction, so that f satisfies (SCD).

Now assume that f satisfies (SIN) and (SCD). Pick $\alpha \in P_m$ and $x \in M$ such that $\alpha_x = 0$. Set $j = \lambda_x(\alpha) - 1$. Since f satisfies (SIN) and $j < x$,

$$\alpha \div f(j) = \frac{1}{f(j)} > \frac{1}{f(x)} = \alpha \div f(x).$$

Thus, $x \notin C_f(\alpha)$ and so C_f satisfies (ME).

Let $z \in C_f(\alpha)$, so that $\alpha_z > 0$ by the above argument. Define

$$\tau_z = \tau_z(\alpha) = \sum_{i=1}^{z} (\alpha_i - \alpha_z) \ge 0,$$

and set

$$g_z(x) = \frac{\tau_z + \alpha_z x}{f(x)} > 0, \qquad \text{for all } x \in M.$$

Note that $\alpha \div f(x) = g_z(x)$ for $x \in [\lambda_z - 1, \rho_z]$. Since f satisfies (SIN) and (SCD), g_z has in M a unique minimum at, say, $\mu_z \in M$; $g_z(x)$ strictly decreases in $[1, \mu_z]$, and it strictly increases in $[\mu_z, m]$. Now, $z \in [\lambda_z, \rho_z]$. If $z \in [\lambda_z, \rho_z)$ and $z \geq \mu_z$, then $g_z(\mu_z) \leq g_z(z) < g_z(\rho_z) \leq g_z(z)$, a contradiction. If $z \in [\lambda_z, \rho_z)$ and $z \leq \mu_z$, then $g_z(\mu_z) \leq g_z(z) \leq g_z(\lambda_z) < g_z(\lambda_z - 1) \leq g_z(z)$, a contradiction. Thus, $z \notin [\lambda_z, \rho_z)$, so that $z = \rho_z(\alpha)$. ∎

COROLLARY 3.2. *If $p \in \Re^{>1}$, then ras_p satisfies (EN) and (ME).*

COROLLARY 3.3. *For $c \in \Re$ and positive integer $m > 1$, let $f : M \to \Re^{>0}$ be such that $f(x) = x^c$ for all $x \in M$, and let $C_f : P_m \to \Pi(M)$ be the consensus rule associated with f. C_f satisfies (EN) and (ME) if and only if $0 < c < 1$.*

Note that (NT) follows from (EN) and (ME), and that the proof of Theorem 3.1 shows that (ME) follows when f satisfies (SIN).

THEOREM 3.4. *Let $f : M \to \Re^{>0}$ and $C_f : P_m \to \Pi(M)$, the consensus rule associated with f. If C_f satisfies (EN) and (ME), then C_f satisfies (NT).*

PROOF. The result is immediate for $m = 1$, so let $m > 1$. Consider $\alpha = (1, 0, \dots, 0) \in P_m$, and $\omega = (m^{-1}, \dots, m^{-1}) \in P_m$. Since C_f satisfies (EN), $\emptyset \neq C_f(\alpha) \subseteq \{1, m\}$ and $C_f(\omega) = \{m\}$. Since C_f satisfies (ME), $m \notin C_f(\alpha)$, so that $C_f(\alpha) = \{1\}$. Since $m > 1$, $C_f(\alpha) = \{1\} \neq \{m\} = C_f(\omega)$, so that $|\{C_f(\alpha) : \alpha \in \bar{P}_m\}| > 1$. ∎

REFERENCES

1. H.-H. Bock, W.H.E. Day and F.R. McMorris, Consensus rules for committee elections, *Mathl. Soc. Sci* **35** (3), 219–232 (1998).
2. H.-H. Bock, W.H.E. Day, E. Kubicka, G. Kubicki and F.R. McMorris, Attainable results in committee elections, *Mathl. Comput. Modelling* **30** (1/2), 75–87 (1999).
3. W.H.E. Day, E. Kubicka, G. Kubicki and F.R. McMorris, The asymptotic plurality rule for molecular sequences, *Mathl. Comput. Modelling* **23** (3), 27–42 (1996).
4. W.H.E. Day and F.R. McMorris, Analysing molecular sequences using consensus, *N. Z. J. Bot.* **31** (3), 211–218 (1993).
5. W.H.E. Day and F.R. McMorris, On the consistency of the plurality rule consensus function for molecular sequences, *J. Classif.* **11** (2), 233–249 (1994).